Ciphering Round Varsity League

High School Math Competition 2006

Georgia Institute of Technology

March 4th, 2006
Problem #1

Problem

Find $a_1 + a_2 + a_3 + \cdots + a_9$ given that

$$(1 + x + x^2)^5 = a_0 + a_1 x + a_2 x^2 + \cdots + a_9 x^9 + a_{10} x^{10}.$$
Problem #1

Problem

Find $a_1 + a_2 + a_3 + \cdots + a_9$ given that

$$(1 + x + x^2)^5 = a_0 + a_1 x + a_2 x^2 + \cdots + a_9 x^9 + a_{10} x^{10}.$$ 

Answer

$$a_1 + a_2 + a_3 + \cdots + a_9 = 3^5 - 2 = 241$$
Problem #2

Problem

Find the prime factorization of $3^{10} + 3^9 - 12$. 
Problem #2

Problem
Find the prime factorization of $3^{10} + 3^9 - 12$.

Answer
$$3^{10} + 3^9 - 12 = 2^7 \cdot 3 \cdot 5 \cdot 41$$
Problem #3

Compute

\[ \frac{1004}{2006} + \left( \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{2005 \cdot 2006} \right). \]
Problem #3

Problem

Compute

\[ \frac{1004}{2006} + \left( \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{2005 \cdot 2006} \right). \]

Answer

\[ \frac{1004}{2006} + \left( \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{2005 \cdot 2006} \right) = 1 \]
Problem #4

Find the sum of the digits of $10^{2005} - 2006$. 
Problem #4

Problem

Find the sum of the digits of $10^{2005} - 2006$.

Answer

18038
Problem #5

If $\alpha$ and $\beta$ are the two roots of the equation $2x^2 + 4x + 3 = 0$, find $\alpha^2 + \beta^2$. 
Problem 

If $\alpha$ and $\beta$ are the two roots of the equation $2x^2 + 4x + 3 = 0$, find $\alpha^2 + \beta^2$.

Answer

$\alpha^2 + \beta^2 = 1$
Problem #6

Problem

If $P \subseteq \{1, 2, 3, \ldots, 48, 49\}$ has the property that it does not have two distinct elements with sum divisible by 7, what is the maximal amount of elements $P$ can have?
Problem #6

Problem

If $P \subseteq \{1, 2, 3, \ldots, 48, 49\}$ has the property that it does not have two distinct elements with sum divisible by 7, what is the maximal amount of elements $P$ can have?

Answer

22
Problem #7

If for all $x \neq 0$ one has that $x^{-1}f(-x) + f(x^{-1}) = x$, find $f(1)$. 

Answer
$f(1) = 1$
Problem #7

Problem
If for all $x \neq 0$ one has that $x^{-1}f(-x) + f(x^{-1}) = x$, find $f(1)$.

Answer
$f(1) = 1$
Problem #8

If \( a > 0 \) and \( a^2 + \frac{1}{a^2} = 7 \), find \( a^3 + \frac{1}{a^3} \).
Problem #8

Problem

If \( a > 0 \) and \( a^2 + \frac{1}{a^2} = 7 \), find \( a^3 + \frac{1}{a^3} \).

Answer

\[ a^3 + \frac{1}{a^3} = 18 \]
Problem \#9

In the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, ..., The ordered sequence with $n$ terms equal to $n$, what is the 2006$^{th}$ term in the sequence?
Problem #9

Problem

In the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5,..., The ordered sequence with $n$ terms equal to $n$, what is the $2006^{th}$ term in the sequence?

Answer

63
Problem #10

Find $a + b + c + d + e$ where each letter represents the measure of the angle indicated in the figure.

**Problem**

$\sum_{i=1}^{5} \theta_i = 180^\circ = \pi$ radians
Problem #10

Problem

Find $a + b + c + d + e$ where each letter represents the measure of the angle indicated in the figure.

Answer

$$a + b + c + d + e = 180^\circ = \pi \text{ radians}$$
THE END