1. The roots of $16(1-x) = \frac{1}{x}$ are $x_1, x_2$. The tenth digit after the decimal point of $x_1$ is 8; find the tenth such digit in $x_2$.

**Solution**: The equation $16(1-x) = \frac{1}{x}$ can be rewritten as

$$16x^2 - 16x + 1 = 16(x - x_1)(x - x_2) = 0,$$

therefore $x_1 + x_2 = 1 = .999999\ldots$ and $x_1x_2 = 1/16$. Then, both $x_1$ and $x_2$ are both positive and therefore the tenth decimal digit after the point of $x_1$ and the tenth decimal digit after the point of $x_2$ have add up to 9. Given this, the tenth digit after the decimal point of $x_1$ is 1.

2. Among all pairs of $2k$ digit natural numbers $a, b$ (in decimal representation) that have the same set of digits with all digits positive, what is the largest possible value of $a - b$? Prove your results. For example, if $n = 2$ then the largest value is $91 - 19 = 72$.

**Solution**: Note that if we give a set of digits $d_0, d_1, \ldots d_{2k-1}$, assuming $1 \leq d_0 \leq d_1 \leq \cdots \leq d_{2k-1} \leq 9$, then, the greatest value would be $a = d_{2k-1}d_{2k-2}\ldots d_1d_0$ and the smallest would be $b = d_0d_1\ldots d_{2k-2}d_{2k-1}$. Thus

$$a - b = (10^{2k-1}d_{2k-1} + \cdots + 10d_1 + d_0) - (10^{2k-1}d_0 + \cdots + 10d_{2k-2} + d_{2k-1})$$

$$= (10^{2k-1} - 1)(d_{2k-1} - d_0) + (10^{2k-2} - 10)(d_{2k-2} - d_1) + \cdots + (10^k - 10^{k-1})(d_k - d_{k-1})$$

Therefore we are trying to maximize the numbers $d_{2k-1} - d_0, d_{2k-2} - d_1\ldots d_{2k-1} - d_2$, what is obtained when $d_0 = d_1 = \cdots = d_{k-1} = 1$ and $d_k = \cdots = d_{2k-2} = d_{2k-1} = 9$, and therefore $a - b = 8888\ldots 8 7 1111\ldots 12$. $k-1$ times $k-1$ times

3. Prove that, for integers $m \geq 1$, $(4m)!/(m!)^4 > ((2m)!)^4$.

**Solution**: Note that

$$\frac{(4m)!/(m!)^4}{(4m)!} > \frac{(2m)!/m!}{m!} \cdot \frac{(2m)!}{m!}$$

Now, suppose that we have a group of $4m$ people, $2m$ of which are males and the other $2m$ are females. There is exactly $\binom{4m}{2m}$ ways of
choosing a subgroup of $2m$ people, and in some of those ways (not all) we will have exactly $m$ males and $m$ females in the subgroup of $2m$. Now, the number of ways of choosing exactly $m$ males and $m$ females would be $\binom{2m}{m}^2$, therefore

$$\binom{4m}{2m} > \binom{2m}{m}^2 \Rightarrow (4m)!(m!)^4 > ((2m)!)^4.$$ 

\[ \square \]

4. 2006 can be factored as $2 \times 17 \times 59$. Prove that for all $x \in \mathbb{R}$

$$\left( \frac{1003}{2} \right)^x + \left( \frac{118}{17} \right)^x + \left( \frac{34}{59} \right)^x \geq 2^x + 17^x + 59^x.$$ 

Remark: $2006 = 1003 \cdot 2 = 118 \cdot 17 = 34 \cdot 59$.

**Solution**: Note that by arithmetic-geometric mean we have that:

$$\frac{1}{2} \left[ \left( \frac{1003}{2} \right)^x + \left( \frac{118}{17} \right)^x \right] \geq \left[ \left( \frac{1003}{2} \right)^x \left( \frac{118}{17} \right)^x \right]^\frac{1}{2} = \frac{59^x}{59^x}$$

Similarly

$$\frac{1}{2} \left[ \left( \frac{118}{17} \right)^x + \left( \frac{34}{59} \right)^x \right] \geq 2^x$$

$$\frac{1}{2} \left[ \left( \frac{1003}{2} \right)^x + \left( \frac{34}{59} \right)^x \right] \geq 17^x$$

Adding all of them together

$$\left( \frac{1003}{2} \right)^x + \left( \frac{118}{17} \right)^x + \left( \frac{34}{59} \right)^x \geq 2^x + 17^x + 59^x.$$ 

\[ \square \]

5. In $\triangle ABC$, $\angle ABC = 30^\circ$, and $\angle ACB = 45^\circ$. Let $E$ be inside of this triangle in such a way that triangle $EBC$ is isosceles (i.e. $|EB| = |EC|$), and $\angle EBC = 15^\circ$. Denote by $F$ the intersection of $CE$ and $AB$. Show that $|AF| = |FB|$.

**Solution**: Note that by Sine Law we have that

$$\frac{|AF| + |FB|}{\sin 45^\circ} = \frac{|AC|}{\sin 30^\circ}$$

$$\frac{|AF|}{\sin 30^\circ} = \frac{|AC|}{\sin 45^\circ}$$

Then, solving the second equation for $|AC|$ and substituting it on the first we have that

$$|AF| + |FB| = \frac{\sin^2 45^\circ}{\sin^2 30^\circ} |AF| = 2|AF|$$

$$\Rightarrow |AF| = |FB|.$$ 

\[ \square \]