Junior Varsity Proof

Solutions

1. Show that at a gathering of any six people, some three of them are either mutual acquaintances or are complete extrangers to each other.

**Solution:** Without loss of generality, assume this people are Alice, Brenda, Charles, David, Ellen and Frank, or $A, B, C, D, E$ and $F$ as a shortcut. From $A$’s standpoint, she either is an acquaintance with at least three people, or is a totally extranger with at least three people. Suppose then, without loss of generality that $A$ is acquaintance with $B, C$ and $D$. Now, if among $B, C$ and $D$, they are all complete extrangers to each other, then the statement is true. If not, then suppose that $B$ and $C$ are mutual acquaintances (symetrically any other pair), then $A, B$ and $C$ are mutual acquaintances, and hence, the statement is true.

2. A regular tetrahedron and a regular octahedron have equal edges. Find the ratio of their volumes.

**Solution:** Let $\ell$ be the length of the edges. A tetrahedron is a pyramid with triangular base, and a octahedron can be seen as two pyramid with square basis glued together by the base. Consider the vertices of the tetrahedron as $X, Y, Z$ and $W$, and let $V$ be the the foot of the height of the tetrahedron over $XYZ$. Now, let $A, B, C, D, E, F$ be the vertices of the octahedron, where $ABCD$ is the square basis. Let $G$ be the foot of the heigt over the square base.

The volume of all piramida figure with straight edges is the area of the base times the height divided by $3$. Let $h_1$ be the height of the tetrahedron and $h_2$ the height of the pyramid $ABCDE$. By symetry $XV$ bisects the angle $\angle XYZ$, $YV$ bisects $\angle XYZ$, hence $V$ is the circumcenter of $\triangle XYZ$, and therefore $XV = \frac{\sqrt{3}}{3} \ell$, and by the Pythagorean theorem one have that $h_1 = \frac{\sqrt{6}}{3} \ell$. Now, $AG$ is half of the diagonal of $ABCD$, hence $AG = \frac{\sqrt{2}}{2} \ell$ and hence $h_2 = \frac{\sqrt{2}}{2} \ell$. Let $\triangle$ be the area of $XYZ$, and
the area of $ABCD$. If $\Lambda_1$ is the volume of the tetrahedron and $\Lambda_2$ be the volume of the octahedron, then,

$$\Lambda_1 = \frac{h_1 \triangle}{3} = \frac{\left( \frac{\sqrt{6}}{3} \ell \right) \left( \frac{\sqrt{3}}{4} \ell^2 \right)}{3} = \frac{\sqrt{2}}{12} \ell^3$$

$$\Lambda_2 = 2 \frac{h_2 \Box}{3} = \frac{2 \left( \frac{\sqrt{2}}{2} \ell \right) \ell^2}{3} = \frac{\sqrt{2}}{3} \ell^3$$

Thus

$$\frac{\Lambda_1}{\Lambda_2} = \frac{1}{4}.$$
4. Solve \[
\begin{align*}
&\begin{cases} 
  a^3 - b^3 - c^3 = 3abc \\
  a^2 = 2(b + c)
\end{cases}
\end{align*}
\] simultaneously in the positive integers.

**Solution:** We will first work the first equation. Note that
\[
3abc = a^3 - b^3 - c^3 = a^3 - (b + c)(b^2 - bc + c^2) = a^3 - (b + c)[(b + c)^2 - 3bc] = [a^3 - (b + c)^3] + 3bc(b + c) = [a - (b + c)][a^2 + a(b + c) + (b + c)^2] + 3bc(b + c)
\]

therefore
\[
0 = [a - (b + c)][a^2 + a(b + c) + (b + c)^2] + 3bc(b + c) - 3abc = [a - (b + c)][a^2 + a(b + c) + (b + c)^2] - 3bc[a - (b + c)] = [a - (b + c)][a^2 + a(b + c) + (b + c)^2 - 3bc] = [a - (b + c)][a^2 + a(b + c) + (b - c)^2 + bc]
\]

Note that the second member of the multiplication is the addition of strictly positive numbers, hence the second bracket is different from zero, and thus,
\[
a - (b + c) = 0 \Rightarrow a = b + c = \frac{a^2}{2} \Rightarrow a = 2 \Rightarrow b = c = 1
\]

What is the only solution.

5. Prove that if \(a, b, c\) are real numbers such that \(a + b + c = 0\), then
\[
3abc = a^3 + b^3 + c^3.
\]
Solution

Note that as $a + b + c = 0$ then we have that

$$
0 = (a + b + c)^3
= a^3 + b^3 + c^3 + 3ab^2 + 3ac^2 + 3ba^2 + 3bc^2 + 3ca^2 + 6abc
= (a^3 + b^3 + c^3) - 3abc + 3(ab^2 + ba^2 + abc)
+ 3(cb^2 + bc^2 + abc) + 3(ac^2 + ca^2 + abc)
= (a^3 + b^3 + c^3) - 3abc + 3ab(a + b + c)
+ 3cb(a + b + c) + 3ac(a + b + c)
= (a^3 + b^3 + c^3) - 3abc
$$

From where evidently $3abc = a^3 + b^3 + c^3$