1. Show that if \( x, y, z, w \) are positive real numbers, then
\[
\frac{(x^2 + x + 1)(y^2 + y + 1)(z^2 + z + 1)(w^2 + w + 1)}{xyzw} \geq 81.
\]

**Solution:** First note that \((x - 1)^2 \geq 0\) for any \(x\), in particular if \(x > 0\). Then \(x^2 + 1 \geq 2x\), and therefore \((x^2 + x + 1) \geq 3x\). The same is true for all the other variables. Then multiplying all four
\[
(x^2 + x + 1)(y^2 + y + 1)(z^2 + z + 1)(w^2 + w + 1) \geq 81xyzw
\]
As all the variables are positive, we can divide by \(xyzw\), and therefore
\[
\frac{(x^2 + x + 1)(y^2 + y + 1)(z^2 + z + 1)(w^2 + w + 1)}{xyzw} \geq 81.
\]

\[\square\]

2. Find the number of paths (that is, moving only vertically or horizontally) in the following array which spell out the word \textit{MATHEMATICIAN}.

```
M
MAM
MATAM
MATHTAM
MATHEHTAM
MATHEMEHTAM
MATHEMAMEHTAM
MATHEMATAMEHTAM
MATHEMATITAMEHTAM
MATHEMATICITAMEHTAM
MATHEMATICICITAMEHTAM
MATHEMATICIANAICITAMEHTAM
```
**Solution:** Consider half of the problem as shown below and notice that all such path have to arrive to the same N.

```
  M
 MA
 MAT
 MATH
 MATHE
 MATHEM
 MATHEMA
 MATHEMAT
 MATHEMATI
 MATHEMATIC
 MATHEMATICI
 MATHEMATICIAN
```

Going backward, you can see that to go from one letter to the previous, there are always two possibilities. Then one has 2 possibilities to go from N to A, two from A to I, and so on. That give us $2^{12}$ possibilities. Now, on the other half we also have $2^{12}$ possibilities, and the only mutual path to both sets is the one that is directly vertical. Hence, there are $2^{13} - 1 = 8191$ different possible paths.

\[ \square \]

3. Show that in every tetrahedron, there must be at least one vertex at which each of the face angles is acute.

**Solution:** First, note that the sum of all the face angles of a tetrahedron is $4\pi$. Now, suppose that $ABCD$ is the tetrahedron. Note that, assuming $\angle BAC$ is the largest of the angles from the vertex $A$, then note that $\angle BAC < \angle BAD + \angle CAD$, and therefore $2\angle BAC < \angle BAC + \angle BAD + \angle CAD$, hence the sum of the angles at any given vertex is strictly larger than twice the largest angle. If every vertex has an angle of at least $\frac{\pi}{2}$, then the sum $S$ of the angles of the tetrahedron would be $S > 4 \cdot 2 \cdot \frac{\pi}{2} = 4\pi$, and this is a contradiction, therefore there is at least one vertex with all its angles acute.

\[ \square \]

4. Prove that if $\alpha, \beta$ and $\gamma$ are the angles of a triangle, then

$$\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$$
Solution: First, note that \( \alpha + \beta + \gamma = \pi \), hence \( \alpha + \beta = \pi - \gamma \), and therefore

\[
-\tan \gamma = \tan(\pi - \gamma) = \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
\]

and therefore

\[
-\tan \gamma + \tan \alpha \tan \beta \tan \gamma = \tan \alpha + \tan \beta \Rightarrow \tan \alpha \tan \beta \tan \gamma = \tan \alpha + \tan \beta + \tan \gamma
\]

\[\square\]

5. The square numbers are numbers of the form \( n^2 \) for some \( n \). The triangular numbers are numbers of the form \( 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \) for some \( n \). Show that there are infinitely many numbers that are both square and triangular.

Solution: First than all, let’s see that we do have a initial case. 1 is both a triangular and a square number. Then, define \( a_1 = 1 \), \( b_1 = 1 \), and define recursively \( a_{n+1} = 4a_n(a_n + 1) \) and \( b_{n+1} = 2b_n(2a_n + 1) \). First note that for any \( n \) one has that \( a_n < a_{n+1} \) and \( b_n < b_{n+1} \), then, both sequence of positive integers are strictly increasing. Now, we will prove that for any \( n \)

\[
\frac{a_n(a_n + 1)}{2} = b_n^2
\]

Note that this is true for \( n = 1 \). Now, suppose it is true for \( n = k \), then, note that

\[
\frac{a_{k+1}(a_{k+1} + 1)}{2} = \frac{4a_k(a_k + 1)[4a_k(a_k + 1) + 1]}{2} = \frac{4a_k(a_k + 1)}{2}(2a_k + 1)^2 = 4b_{k+1}^2(2a_k + 1)^2 = [2b_k(2a_k + 1)]^2 = b_{k+1}^2
\]

Hence, we have two increasing infinite sequences \( a_n \) and \( b_n \) such that the triangular number of the first equals the square of the other, hence, there are infinitely many numbers that are both square and triangular.

\[\square\]