Is there any pair of positive integers \(a\) and \(b\) such that the number

\[(2a + b)(2b + a)\]

is a power of 2? Justify your answer.

**Solution:** Let \(2^m\) be the maximum power of 2 dividing \(a\) and \(2^n\) the maximum power of 2 dividing \(b\). Given the symmetry of the problem, we may assume without any loss of generality that \(m \leq n\). Call \(\hat{a}\) and \(\hat{b}\) the positive integer numbers such that \(a = 2^m\hat{a}\) and \(b = 2^n\hat{b}\). Note that \(\hat{a}\) has to be an odd number. Now

\[(2a + b)(2b + a) = 2^m(2a + b)(2\hat{b} + \hat{a}),\]

But \(2\hat{b} + \hat{a} \geq 3\) and it’s an odd number, thus \((2a + b)(2b + a)\) has at least one odd prime factor, and by uniqueness of the prime factorization, such number cannot be a power of 2, regardless of what pair \((a, b)\) is chosen.

\(\square\)
Let $a$, $b$, $c$ and $d$ distinct real numbers such that $a > 0$, the equation $x^2 - 3ax - 8b = 0$ has roots $c$ and $d$, and the equation $x^2 - 3cx - 8d = 0$ has roots $a$ and $b$. Find the value of the sum $a + b + c + d$.

**Solution:** Note that under these conditions, none of the numbers $a$, $b$, $c$ or $d$ can be 0. If $a = 0$ or $b = 0$, then 0 is a root of $x^2 - 3cx - 8d = 0$, and hence $d = 0$, but we know all the numbers are different, so, this cannot be the case. In the same way neither $d$ nor $c$ can be 0. Now, $x^2 - 3ax - 8b = (x - c)(x - d)$ and $x^2 - 3cx - 8d = (x - a)(x - b)$, hence

\[
\begin{align*}
a + b &= 3c \\
c + d &= 3a
\end{align*}
\]

Adding the two equations on the left, one has that $a + b + c + d = 3(a + c)$ or equivalently $b + d = 2(a + c)$. On the other hand, multiplying the two equations on the right one gets that $abcd = 64bd$, and as none of the numbers is zero, one can simplify to $ac = 64$. Now, if one evaluates the first polynomial in $a$ and the second in $c$, one would get that

\[
\begin{align*}
c^2 - 3ac - 8b &= 0 \\
a^2 - 3ac - 8d &= 0
\end{align*}
\]

Adding these two equalities, one gets

\[
\begin{align*}
0 &= (a^2 + c^2) - 6ac - 8(b + d) \\
&= (a^2 + 2ac + c^2) - 16(a + c) - 8ac \\
&= (a + c)^2 - 16(a + c) - 512 \\
&= [(a + c) + 16][(a + c) - 32]
\end{align*}
\]

Note that as $a > 0$ and $ac = 64$, then $c$ is also positive, and so is $a + c$. Therefore $a + c = 32$ and hence $a + b + c + d = 96$. \qed
Define

\[ n!! = n(n - 2)(n - 4)(n - 6) \cdots = \prod_{k=0}^{\frac{n-1}{2}} (n - 2k). \]

Prove the identity

\[ \frac{2n-3)!!2^n}{n!} = \frac{1}{2n-1} \binom{2n}{n} \]

**Solution:**

\[
\begin{align*}
\frac{1}{2n-1} \binom{2n}{n} &= \frac{1}{2n-1} \cdot \frac{(2n)!}{(n!)(n!)} \\
&= \frac{1}{2n-1} \cdot \frac{(2n-1)!!(2n)(2n-2)(2n-4) \cdots 2}{(n!)(n!)} \\
&= \frac{1}{2n-1} \cdot \frac{(2n-1)!!2^n(n)(n-1)(n-2) \cdots 1}{(n!)(n!)} \\
&= \frac{1}{2n-1} \cdot \frac{(2n-1)!!2^n}{(n!)(n!)} \\
&= \frac{1}{2n-1} \cdot \frac{(2n-1)!!2^n}{n!} \\
&= \frac{(2n-3)!!2^n}{n!}
\end{align*}
\]

\(\square\)
Problem #4

Prove that for every positive integer \( n \), there is a polynomial \( p_n(x) \) with integer coefficients and degree \( n \) such that for every real number \( \theta \), \( \cos(n\theta) = p_n(\cos \theta) \).

**Solution:** For this proof we will strongly use the identity \( \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \). Note that

\[
\begin{align*}
\cos((n+1)\theta) &= \cos(n\theta) \cos \theta - \sin(n\theta) \sin \theta \\
\cos((n-1)\theta) &= \cos(n\theta) \cos \theta + \sin(n\theta) \sin \theta.
\end{align*}
\]

and adding these two identities one gets

\[
\cos((n+1)\theta) = 2 \cos(n\theta) \cos \theta - \cos((n-1)\theta).
\]

Now, define \( P_1(x) = x, P_2(x) = 2x^2 - 1 \), and for \( n \geq 2 \) define \( P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x) \). By induction, as \( P_1(x) \) and \( P_2(x) \) have integer coefficients, and if \( P_n(x) \) and \( P_{n-1}(x) \) then \( P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x) \) has integer coefficients, then, what is left to prove is that for every \( n \) it holds that \( \cos(n\theta) = P_n(\cos \theta) \).

Note that \( P_1(\cos \theta) = \cos \theta \) and \( P_2(\cos \theta) = 2\cos^2 \theta - 1 = \cos 2\theta \). Now, by induction, assume that \( P_n(\cos \theta) = \cos(n\theta) \) and that \( P_{n-1}(\cos \theta) = \cos((n-1)\theta) \). Then, as \( P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x) \), we have that

\[
\begin{align*}
P_{n+1}(\cos \theta) &= 2 \cos \theta P_n(\cos \theta) + P_{n-1}(\cos \theta) \\
&= 2 \cos(n\theta) \cos \theta - \cos((n-1)\theta) \\
&= \cos((n+1)\theta).
\end{align*}
\]

These are known as Chebyshev polynomials of the first kind.
Let $A, B, C$ and $D$ be four different points in the plane such that the angles $\angle ACD$ and $\angle BCD$ are obtuse. Prove that for any point $E$ with $CE \leq CD$ the following inequality holds:

$$2AD \cdot BD > (AC + CD) \cdot BE.$$ 

**Solution:** Note that as $\angle ACD$ and $\angle BCD$ are obtuse, then

$$AD^2 > AC^2 + CD^2 \geq \frac{(AC + CD)^2}{2} \Rightarrow \sqrt{2AD} > (AC + CD),$$

$$BD^2 > BC^2 + CD^2 \geq \frac{(BC + CD)^2}{2} \Rightarrow \sqrt{2BD} > (BC + CD).$$

Multiplying these two inequalities, one gets that

$$2AD \cdot BD > (AC + CD) \cdot (BC + CD).$$

Now, as $CD \geq CE$, then $BC + CD \geq BC + CE \geq BE$ by triangle inequality, and thus

$$2AD \cdot BD > (AC + CD) \cdot BE.$$