3 Proof

1. Find all real numbers $u$ so that the equation $(u + 1)x^2 - (u^2 - u)x + (u - 2) = 0$ does not have two distinct real roots in $x$.

Solution: When $u = -1$, the equation becomes $-2x - 3 = 0$ and it has only one real root. Otherwise consider discriminant, by writing $w = (u + 1)(u - 2) = u^2 - u - 2$, we have $\Delta = (u^2 - u)^2 - 4(u + 1)(u - 2) = w^2 - 4w = w^2 + 4 > 0$, so the discriminant is always positive and the equation would have two real roots. Therefore the only possible value of $u$ is $-1$.

2. In a round robin tournament there are $N$ players $A_1, \ldots, A_N$. Any two players play against each other exactly once, each win (resp. draw, loss) is awarded 2 points (resp. 1, 0 points). It turns out that after the tournament the scores of $A_1, \ldots, A_n$ are strictly decreasing in this order, but $A_i+1$ defeated $A_i$ for $i = 1, 2, \ldots, N - 1$, and the match between $A_1$ and $A_N$ ended in a draw. Find the minimum possible value of $N$.

Solution: $A_1$ can have at most $2N - 5$ points and $A_N$ has at least 3 points, but the two players differ by at least $N - 1$ points, so $2N - 5 \geq 3 + (N - 1)$, or $N \geq 7$. $N = 7$ is possible: beside the given results, set the matches between $A_2 - A_4, A_4 - A_6, A_2 - A_6, A_3 - A_5$ to be draws and in all other matches the smaller index player beats the bigger index one, it is easy to check the scores of $A_1, A_2, \ldots, A_N$ are 9, 8, \ldots, 3 respectively.

3. Find all integers $n \geq 2$ such that the number $11111_n$ (in base $n$) is a perfect square.

Solution: The number $11111_n$ is equal to $n^4 + n^3 + n^2 + n + 1$. Now notice that $(n^2)^2 = n^4 < n^4 + n^3 + n^2 + n + 1 < n^4 + 2n^3 + n^2 = (n^2 + n)^2$. Hence we must have $n^4 + n^3 + n^2 + n + 1 = (n^2 + k)^2$ for some integer $k$ between 1 and $n - 1 < \sqrt{n^2}$ inclusive. If we divide $n^4 + n^3 + n^2 + n + 1$ by $n^2$, the quotient will be $n^2 + n + 1$, and the remainder will be $n + 1$. If we divide $(n^2 + k)^2 = n^4 + 2kn^2 + k^2$ by $n^2$, the quotient will be $n^2 + 2k$ and the remainder will be $k^2$. Hence, we have $n^2 + n + 1 = n^2 + 2k$, i.e. $n + 1 = 2k$, and $n + 1 = k^2$. Therefore $2k = k^2$, and so we must have $k = 2$. Indeed, when $n = 2k - 1 = 3$, $11111_3 = 121_{10}$ is a perfect square.

4. An infinite sequence $a_1, a_2, a_3, \ldots$ of 1’s and 2’s is uniquely defined by the following properties:

1. $a_1 = 1$ and $a_2 = 2$. 

2. For every $n \geq 1$, the number of 1’s between the $n^{th}$ 2 and the $(n + 1)^{st}$ 2 is equal to $a_{n+1}$.

Is the sequence periodic from the beginning?

**Solution:** No. Suppose for a contradiction that the sequence has period $k$, where $k$ is chosen to be as small as possible. Consider first the case $a_k = 1$. Then the sequence can be grouped into repeated blocks as follows:

$$12112\ldots1 12112\ldots1 12112\ldots1\ldots$$

Hence the substring 112112 appears in the sequence. So we must have $a_i = a_{i+1} = 2$ for some $i$, but that is impossible as there are no 1’s between the two 2’s here.

Now consider the case $a_k = 2$. The sequence therefore looks like

$$12112\ldots2 12112\ldots2 12112\ldots2\ldots$$

By definition, the $n^{th}$ 2 has exactly $a_n$ 1’s preceding it, so replacing each 12 with a 1 and each 112 with a 2 yields back the original sequence. But the disjoint repeated blocks remain disjoint and identical after such replacement procedure. This gives a smaller period and a contradiction to the minimality of $k$.

5. Let $\triangle ABC$ be a triangle. Let $D$ be a point on $BC$ such that $AD \perp BC$. Let $E$ and $F$ be points on $AB$ and $AC$ respectively such that $DE \perp AB$ and $DF \perp AC$. Extend $EF$ to meet the circumcircle of $\triangle ABC$ at $X$ and $Y$ respectively ($X$ is on the arc $AB$ and $Y$ is on the arc $AC$). (a) Show that $AX = AD = AY$; (b) show that $\sin \angle A = \frac{XC}{AC} - \frac{XB}{AB}$.

You can use the result of (a) for (b) even if you can not solve (a).

**Solution:** (a) Say $XY$ meets $AO$ at $Z$. We have $\angle AEZ = \angle ADF = \angle C$, the first equality follows from that $A, E, D, F$ are concyclic; also $\angle ZAE = 90^\circ - \frac{1}{2} \angle AOB = 90^\circ - \angle C$, so $\angle AZE = 90^\circ$ and $AO \perp XY$, in particular $AX = AY$. Now $\angle AXE = \angle AYX = \angle ABX$, hence $\triangle AXE \sim \triangle ABX$; we also have $\triangle ADE \sim \triangle ABD$, thus $AX^2 = AE \cdot AB = AD^2$, that is $AX = AD$.

(b) By Ptolemy theorem and basic area formulae we have $XC \cdot AB - XB \cdot AC = XA \cdot BC = AD \cdot BC = 2S_{ABC} = AB \cdot AC \sin \angle A$, now divide both sides by $AB \cdot AC$.

Tiebreaker. Let $S$ be a set on 2016 elements. Let $\ast$ be a binary operation on $S$, that is, for any $a, b \in S$, we have that $a \ast b$ is an element of $S$. Suppose that $(a \ast b) \ast c = a \ast (b \ast c)$. Suppose further that $a \ast b = b \ast a$ if and only if $a = b$. Show that for any $a, b, c \in S$, $(a \ast b) \ast c = a \ast c$ for any $a, b, c$ from $S$. (Partial credit will be given for any non-trivial progress toward this identity.)
Solution: First $(a*a)°a = a°(a°a)$ implies $a°a = a$ for any $a \in S$; then $(a°b°a)°a = a°b°a = a°(a°b°a)$ implies $a°b°a = a$ for any $a, b \in S$; and finally $(a°b°c)°(a°c) = (a°b°c°a)°c = a°c = a°(c°a°b°c) = (a°c)°(a°b°c)$ implies $a°b°c = a°c$.

End of exam.